A Bethe Ansatz Study of Free Energy and Excitation Spectrum for even spin Fateev Zamolodchikov Model

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Abstract

A Bethe Ansatz study of a self dual Z_N spin model is undertaken for even spin system. One has to solve a coupled system of Bethe Ansatz Equations (BAE) involving zeroes of two families of transfer matrices. A numerical study on finite size lattices is done for identification of elementary excitations over the Ferromagnetic and Antiferromagnetic ground states. The free energies for both Ferromagnetic and Antiferromagnetic ground states and dispersion relation for elementary excitations are found.

I. Introduction

The present model was first proposed in 1982 by V.A. Fateev and A.B. Zamolodchikov [1] as a two dimensional self dual Z_N lattice model with nearest neighbor spin-spin interaction. Baxter, Bazhanov and Perk [2] discovered a set of functional equations involving families of Chiral Potts (CP) transfer matrices. Fateev Zamolodchikov model (FZM) was shown to be a non-chiral self dual limit of Chiral Potts [3, 4]. The Chiral Potts transfer matrix functional equations were used to obtain transcendental equations (Bethe Ansatz Equations) [5] for the zeroes of the transfer matrices for the present problem [3, 4]. In the general FZM model the Bethe Ansatz Equations (BAE) are coupled involving two automorphically connected families of transfer matrices as in the CP case [2]. In the odd spin case these families are connected by simple transformations and the BAEs greatly simplify and decouple requiring us to solve only one set of equations [4]. For this odd spin case alone, Albertini obtained the ferromagnetic ground state [6]. A unified treatment for the ground state of odd and even spin FZM can be found in a previous work [4]. For $N = 4$ FZM a comprehensive study has been done. This includes determination of exact energy values and central charge [7], and completeness and classification of Bethe states [8]. The present work demonstrates the study of these Bethe Equations in both ferromagnetic and antiferromagnetic cases for finding the free energy and elementary excitations.

In a generic situation for Fateev Zamolodchikov model (to be specific we shall use the even spin case) we obtained coupled Bethe Ansatz equations,

$$\prod_{k=1}^{L_{Uq}} \frac{\sinh(\lambda_j - \bar{\lambda}_k - i\gamma)}{\sinh(\lambda_j - \bar{\lambda}_k + i\gamma)} = (-1)^{M+1} \left[\frac{\sinh 2(\lambda_j + is\gamma)}{\sinh 2(\lambda_j - is\gamma)}\right]^{2M} \quad (1)$$

$$\prod_{k=1}^{L_q} \frac{\sinh(\bar{\lambda}_j - \lambda_k - i\gamma)}{\sinh(\bar{\lambda}_j - \lambda_k + i\gamma)} = (-1)^{M+1} \quad (2)$$

where $\{\lambda_j\}$ and $\{\bar{\lambda}_j\}$ are the spectral variables for transfer matrices T_q and T_{U_q} respectively and $\gamma = \frac{\pi}{2N}$ and $s = \frac{1}{2}$.

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The first equation is quite similar to the generic case of Bethe Equation. However it involves zeroes of two different transfer matrices ($\lambda,\ \bar{\lambda}$) that are coupled further by a second equation whose form is quite unique.

The generic case of most commonly encountered Bethe Ansatz Equation looks like

$$\prod_{k=1}^{L} \frac{\sinh(\lambda_j - \lambda_k - i\gamma)}{\sinh(\lambda_j - \lambda_k + i\gamma)} = (-)^{M+1} \left[\frac{\sinh(\lambda_j - iS\gamma)}{\sinh(\lambda_j + iS\gamma)} \right]^{2M}$$

where S is the spin and γ is the anisotropy parameter of the model. In Bethe's original paper [5], he studied the case $\gamma \to \infty$, where the hyperbolic functions reduce to rational ones. Note that in the generic case we have only one type of λ_j as opposed to two species of λ_j and $\bar{\lambda}_j$ as they appear in the present problem.

The standard procedure of calculating the physical quantities, e.g., the energy spectrum, dispersion curves, free energy, is to assume that the solutions for the Bethe Equation are given by the string hypothesis. Starting with the work of Bethe there has been a great deal of study in these complex solutions of BAE. They appear in the form,

$$\lambda_{\alpha,k}^{(n,\nu)} = \lambda_{\alpha}^{(n,\nu)} + \frac{\gamma}{2}(n+1-2k)i + \frac{(1-\nu)\pi}{4} i + \delta_{\alpha,k}^{(n,\nu)} \quad k = 1, 2 \ldots n$$

where $\lambda_{\alpha}^{(n,\nu)}$ is the real part, n is its length, k runs from 0 through n labelling the root. The coefficient ν takes on the value $(+1)$ (positive parity) or (-1) (negative parity); $\delta_{\alpha,k}^{(n,\nu)}$ vanishes regularly as $M \to \infty$.

However Bethe himself realized that for large M not all solutions of BAE are of the form (above) with $lim_{M\to\infty} Im(\delta) = 0$. Modern work for the case of $lim_{M\to\infty} Im(\delta) \neq 0$ was initiated by Destri and Lowenstein and Woynarovich who introduced the definition of narrow pairs for $lim_{M\to\infty} Im(\delta) < 0$ and wide pairs for $lim_{M\to\infty} Im(\delta) > 0$ and was furthered by Avdeev and Dörfel [9, 10] who introduced 3 classes for $lim_{M\to\infty} Im(\delta)$. It is clear that none of the existing approaches to BAE for S integer or half integer is sufficiently refined to answer the reality or completeness question for the Hamiltonian.

In the general spin hypothesis framework one obtains equations for the centers of the strings by multiplying the Bethe Ansatz equations over different members of the same string and then taking the logarithm of the resulting equation. This yields

$$\frac{1}{2\pi} \Theta_j^{(1)}(\lambda_{\alpha}^{j}) - \frac{1}{2\pi M} \sum_{k} \sum_{\beta=1}^{M^{(k)}} \Theta_{jk}^{(2)}(\lambda_{\alpha}^{(j)} - \lambda_{\beta}^{(k)}) = \frac{I_{\alpha}^{(j)}}{M}$$

where $M^{(k)}$ is the number of k-strings and

$$\Theta_j^{(1)}(\lambda) = 2 \sum_{\ell=1}^{n_j} \phi(\lambda, n_j + 2s - 2\ell + 1, \nu_j)$$

$$\Theta_{jk}(\lambda) = \phi(\lambda, n_j + n_k, \nu_j\nu_k) + \phi(\lambda, |n_j - n_k|, \nu_j\nu_k) + \sum_{\ell=1}^{\min(n_j,n_k)-1} 2\phi(\lambda, |n_j - n_k| + 2\ell, \nu_j\nu_k)$$

and

$$\phi(\lambda, n, \nu) = \begin{cases} 2\nu \arctan(\cot(\frac{n\gamma}{2})^{\nu} \tanh(\lambda)) \\ 0 \text{ if } n\gamma = q\pi \quad q \in Z \end{cases}$$

In certain cases, e.g., δ-function Bose gas, it can be proved that the solutions are real, and no such multiplying of string components is necessary [11]. In such cases, the integers designating the branches of logarithms uniquely characterize the states and may be viewed as quantum numbers for the states. A monotonic relation is shown to exist between the integers and the values of the spectral variable $\lambda_\alpha^{(n,v)}$. In almost all subsequent work in the field this unique characterization of states by integers and their monotonic relation to solutions have been assumed. In few cases some counting argument is attempted [5, 12, 13, 14, 15] to justify this assumption. In the $N \to \infty$ limit, after introducing the concept of density of string centers, one obtains a coupled set of integral equations. These equations are manipulated to calculate the energies of the ground state and low lying excited states. For the spin 4 system, complete classification of Bethe states, and exact calculation for finite and infinite systems is already known [7, 8].

In the present problem, we had to deal with a doubly coupled set of integral equations; two sets of coupled equations both involving zeroes of two types of transfer matrices. However linearity of the equations made it possible to solve for the ground state and elementary excitations by Fourier transform method. A study of the classification of roots is undertaken. However one has to keep track of the added complexity of having to handle T_q and T_{U_q} simultaneously. One can identify the ground state and elementary excitations on the basis of this numerical study. The excitation spectrum and dispersion relations can hence be calculated.

II. Fateev-Zamolodchikov model

V.A. Fateev and A.B. Zamolodchikov proposed in 1982 a two dimensional self dual Z_N lattice spin model with nearest neighbor interaction. They obtained this model as the self dual [16] solution of the star-triangle relations [17].

A general Z_N model can be defined as follows. On a two dimensional rectangular lattice the lattice sites are occupied by a spin variable z which takes its values in the group Z_N $[z^N = 1]$. If one designates the sites on the lattice by a two dimensional integer-valued vector x, one can write down the partition function of the statistical Z_N model with nearest neighbor interaction as:

$$Z = \sum_{\{z\}} \prod_{\boldsymbol{x}} \prod_{\sigma=\pm} w^{(\sigma)}(z(\boldsymbol{x}), z(\boldsymbol{x} + \boldsymbol{\epsilon}_\delta)). \tag{3}$$

where the sum runs over all values of the variable z in every site of the lattice. The functions w^σ, $(\sigma = \pm 1)$ are the weight functions corresponding to the interaction between spins on the neighboring sites of the lattice in horizontal $(\sigma = 1)$ and vertical $(\sigma = -1)$ directions respectively. The vectors $\epsilon_1 = (1, 0)$ and $\epsilon_{-1} = (0, 1)$ are the basis vectors of the lattice.

In the absence of external fields, the most general interaction between two neighboring spins after appropriate normalization is given by

$$w^{(\sigma)}(z_1, z_2) = 1 + \sum_{i=1}^{N-1} x_i^{(\sigma)} \cdot (z_1 z_2^\star)^i \tag{4}$$

where superscript * denotes complex conjugate. Reality of $w^{(\sigma)}(z_1, z_2)$ imposes on the parameters the following restriction

$$x_i^{(\sigma)} = x_{N-i}^{(\sigma)} \tag{5}$$

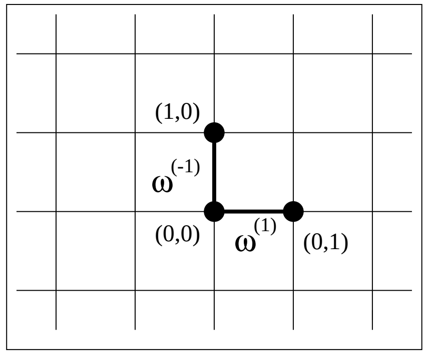

Figure 1: Fateev Zamolodchikov Model on a square lattice

The dual transformation of the statistical weights are given by

$$\tilde{x}_i^{(\sigma)} = \left(1 + \sum_{k=1}^{N-1} x_k^{(-\sigma)} \omega^{ki}\right) \cdot \left(1 + \sum_{k=1}^{N-1} x_k^{(-\sigma)}\right)^{-1} \tag{6}$$

where $\omega = exp(2\pi i/N)$. The region of self duality is then given by

$$\tilde{x}_i^{(\sigma)} = x_i^{(\sigma)} \tag{7}$$

Let the parameters $x_i^{(\sigma)}$ be represented by a family of functions $W_i(\alpha)$ of auxiliary parameter $\alpha \ \in \mathcal{C}$

$$x_i^{(1)} = W_i(\alpha), \quad x_i^{(-1)} = W_i(\pi - \alpha). \tag{8}$$

The star-triangle relation [18, 19] on $x_i(\alpha)$:

$$\begin{aligned} &\sum_{k=0}^{N-1} W_{n_1-k}(\alpha) W_{n_2-k}(\pi - \alpha - \alpha') W_{n_3-k}(\alpha') = \\ &c(\alpha, \alpha') W_{n_2-n_3}(\pi - \alpha) W_{n_1-n_3}(\alpha + \alpha') W_{n_1-n_2}(\pi - \alpha') \end{aligned} \tag{9}$$

The particular solution of Eq. (9) that possesses the self duality property, e.g. Eq. (7), is given by:

$$W_0 = 1, \quad W_n(\alpha) = \prod_{k=0}^{n-1} \frac{\sin[\pi k/N + \alpha/2N]}{\sin[\pi(k+1)/N - \alpha/2N]}. \tag{10}$$

Denoting $x_n^{(1)} = W(n \mid u)$ and $x_n^{(-1)} = \overline{W}(n \mid u)$ we get

$$\frac{W(n|u)}{W(0|u)} = \prod_{j=1}^{n} \frac{\sin(\pi j/N - \pi/2N - u)}{\sin(\pi j/N - \pi/2N + u)} \tag{11}$$

$$\frac{\overline{W}(n|u)}{\overline{W}(0|u)} = \prod_{j=1}^{n} \frac{\sin(\pi j/N - \pi/N + u)}{\sin(\pi j/N - u)} \tag{12}$$

We adopt the normalization $W(0|u) = \overline{W}(0|u) = 1$. The "physical region" defined by non-negative real Boltzmann Weights (BW) , is given by $u \in [0, \pi/2N[$. For $N = 2, 3$ Eq. (11) and Eq. (12) simply reduce to the self-dual critical Potts model. For $N = 4$ it gives a particular case of critical Ashkin-Teller model. Fateev and Zamolodchikov propose that for $N = 5, 7$ the solution describes the critical bifurcation points in the phase diagram of Alcaraz and Koberle [20].

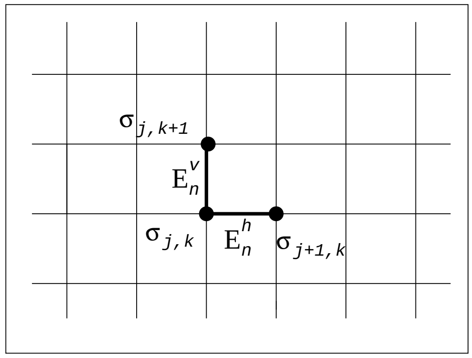

Figure 2: Square lattice Chiral Potts Model

III. Chiral Potts model and connection to FZM

On the sites of a two dimensional lattice of size $\mathcal{M} \times \mathcal{N}$ denoted by two dimensional vector (j,k) with integer entries, we place Z_N spins $\sigma_{j,k}$. The spins $\sigma_{j,k}$ are classical variables satisfying:

$$\sigma_{j,k}^N = 1$$

i.e., $\sigma_{j,k} = \omega^\nu, \nu \in \{0,1,2,\ldots N-1\}$ where ω is the complex N^{th} root of unity with the minimum argument.

$$\omega = e^{2\pi i/N}$$

The energy corresponding to a given configuration of spins $\{\sigma_{j,k}\}$ is

$$\mathcal{E} = -\sum_{\{j,k\}} \sum_{n=1}^{N-1} \{E_n^h \cdot (\sigma_{j,k}\sigma_{j,k+1}^\star)^n + E_n^v \cdot (\sigma_{j,k}\sigma_{j+1,k}^\star)^n\}.$$

Row index j runs over 1 to $\mathcal{M}$ and column index k runs over 1 to $\mathcal{N}$ with periodic boundary condition in both directions implied.

In Chiral Potts a subspace of the coupling parameters (E_n^h, E_n^v) is chosen which has a built-in handedness, or phase. The energy of a nearest neighbor pair is chosen as

$$\mathcal{E}_{pair}^{h,v}(\sigma_1,\sigma_2) = -\sum_{n=1}^{N-1} E_n^{h,v} \cdot (\sigma_1\sigma_2^\star)^n$$

where $E_n^{h,v} = |E_n^{h,v}| \cdot e^{i\delta_n}$. The local Boltzmann weights (BW) can now be easily defined as:

$$W^{h,v}(n) = e^{\frac{1}{k_B T}\sum_{j=1}^{N-1} E_j^{h,v}\omega^{jn}}$$

Let us denote two adjacent row configurations by $\{l\}$ and $\{l'\}$ (l corresponds to the lower row), where

$$\{l\} = \{\omega^{l_j} \mid j = 1(1)\mathcal{N} \;\; and \;\; l_j \in 0,1,\ldots,N-1\}.$$

The row to row transfer matrix is given by:

$$T_{\{l\},\{l'\}} = \prod_{j=1}^{\mathcal{N}} W^h(l_j - l'_{j+1}) \cdot W^v(l_j - l'_j).$$

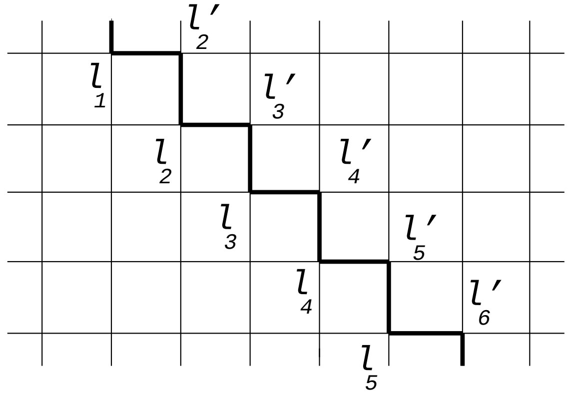

Figure 3: Transfer Matrix $T_{\{l\},\{l'\}}$

It has been shown by several authors [1, 21, 22, 23] that the transfer matrices corresponding to the interaction parameters belonging to the Chiral Potts submanifold commute.

$$[T, T'] = 0$$

The self dual Chiral Potts model is given by BWs

$$\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{j=1}^{n} \frac{b_q - \omega^j a_p}{b_p - \omega^j a_q}. \tag{13}$$

$$\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{j=1}^{n} \frac{\omega a_p - \omega^j a_q}{b_q - \omega^j b_p}. \tag{14}$$

where $\omega = \exp(2\pi i/N)$ and the paired complex variables $(a, b) \in \mathcal{C}^2$ satisfy the constraint

$$a_x^N + b_x^N = \kappa \tag{15}$$

$\kappa \in [0, 1]$, and $x = p$ or q. In the non chiral limit, when $\kappa = 0$, we can parametrize (a_x, b_x) in Eq. (15) as:

$$a_x = e^{2ix} \quad b_x = \omega^{1/2} e^{2ix}. \tag{16}$$

Defining $u = q - p$ Eq. (13) and Eq. (14) reduce to Eq. (11) and Eq. (12). However we retain suffixes (p, q) in the Boltzmann weights $W_{pq}(n|u)$ and $\overline{W}_{pq}(n|u)$ to signify that these BWs are obtained from the Chiral Potts BWs defined in terms of p and q variables.

The transfer matrix for the FZM can be constructed from the BWs as:

$$T_{p,q}^{\boldsymbol{n},\boldsymbol{n}'}(u) = \prod_{k=1}^{M} \overline{W}_{pq}(n_k - n'_k|u) W_{pq}(n_k - n'_{k+1}|u). \tag{17}$$

where M is the number of sites in each row and periodic boundary condition is implied. These transfer matrices for different spectral variable u form a commuting set. This can be argued from the fact that these transfer matrices come as a limit of CP transfer matrix, which are known to be commuting. A more direct argument would be that Fateev and Zamolodchikov obtained FZM BWs as solutions of star-triangle relation, and hence the transfer matrix constructed out of them ought to commute.

$$[T(u), T(u')] = 0 \quad \forall \; u, u' \in C \tag{18}$$

Transfer matrix $T(u)$ reduces to identity operator for $u \to 0$. An expansion of $T(u)$ gives us the associated spin chain Hamiltonian H.

$$T(u) = 1 - Mu \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} - uH + O(u^2) \tag{19}$$

$$H = -\sum_{k=1}^{M} \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} (X_k^n + Z_k^n Z_{k+1}^{-n}). \tag{20}$$

where X and Z are defined as

$$\begin{aligned} X_k|n_1 \dots n_k \dots n_M > &= |n_1 \dots n_k + 1 \dots n_M > \mathrm{mod} N \\ Z_k|n_1 \dots n_k \dots n_M > &= \omega^{n_k}|n_1 \dots n_k \dots n_M > \quad . \end{aligned}$$

Eq. (19) and Eq. (20) imply that each Hamiltonian commutes with all the transfer matrices and their associated Hamiltonians. Thus it has an infinite set of conserved charges in involution. However only a subset of them, whose number is equal to the degrees of freedom of the system, are independent.

In order to obtain the zeros of the eigenvalues of the transfer matrix T_q, we will use functional equations connecting T_q with its automorphically conjugate partners. Thus it is important to understand the relevant automorphisms of the constraint Eq. (15). It has been claimed in the previous section that the transfer matrices constructed out of CP BWs, Eq. (13) and Eq. (14). commute as long as they satisfy Eq. (15). For any $(a, b) \in \mathcal{C}^2$ satisfying the above relations there exist other complex pairs connected to them which satisfy the same relation. Two such automorphic relations of importance are,

$$R(a, b) = (b, \omega a), \qquad U(a, b) = (\omega a, b) \tag{21}$$

It is rather straightforward to check;

$$a_{Rx}^N + b_{Rx}^N = \kappa, \qquad a_{Ux}^N + b_{Ux}^N = \kappa$$

from the relation

$$a_x^N + b_x^N = \kappa$$

If one makes an attempt to go over from CP BWs to FZM BWs through a limiting process, one gets the following relations for W_{pRq}, $\overline{W}_{pRq}$, W_{pUq} and $\overline{W}_{pUq}$,

$$\frac{W_{pRq}(n|u)}{W_{pRq}(0|u)} = \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/N - u)}{\sin(\pi k/N + u)} \tag{22}$$

$$\frac{\overline{W}_{pRq}(n|u)}{\overline{W}_{pRq}(0|u)} = \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/2N + u)}{\sin(\pi k/N - \pi/2N - u)} \tag{23}$$

$$\frac{W_{pUq}(n|u)}{W_{pUq}(0|u)} = e^{-\frac{\imath\pi n}{N}} \prod_{k=1}^{n} \frac{\sin(\pi k/N - \pi/2N - u)}{\sin(\pi k/N + \pi/2N + u)} \tag{24}$$

$$\frac{\overline{W}_{pUq}(n|u)}{\overline{W}_{pUq}(0|u)} = e^{\frac{\imath\pi n}{N}} \prod_{k=1}^{n} \frac{\sin(\pi k/N + u)}{\sin(\pi k/N - u)} \tag{25}$$

Thus in the non chiral limit, $T_q \longrightarrow T_q(u)$ and $T_{Rq} \longrightarrow T_q(u+\pi/2N)$. There is no simple relation between T_q and T_{Uq} though. However, we do feel that there must exist some nontrivial mapping between T_q and T_{Uq} whose understanding will unravel the connection between the zeroes of T_q and T_{U_q} and will give the satisfactory derivation of completeness of states.

IV. Bethe Ansatz type equations for the even N FZM

We define the normalized transfer matrices by removing their denominators,

$$T_q^N(u) = [g_q(u)\overline{g}_q(u)]^M T_q(u)$$

where

$$g_q(u) = \prod_{j=1}^{N/2} \sin(\frac{\pi j}{N} - \frac{\pi}{2N} + u) \quad \overline{g}_q(u) = \prod_{j=1}^{N/2} \sin(\frac{\pi j}{N} - u)$$

One must note that the superscript in T_q^N denotes "normalize" and is not related to the spin quantum number N. Each entry of $T_q^N(u)$ is a product of NM sines and it has the general form

$$\prod_{k=1}^{NM} (c_k^{(1)} e^{iu} + c_k^{(2)} e^{-iu})$$

The calculation of this section goes in the same spirit as that of odd N case. Hence we only quote the results.

$$\begin{aligned} \Lambda_{Q=0}(u) &= \left[\frac{g_q(0)\overline{g}_q(0)}{g_q(u)\overline{g}_q(u)}\right]^M \prod_{k=1}^{L} \frac{\sin(u - v_k)}{\sin v_k} \\ L &= A + B = NM \end{aligned} \tag{26}$$

The normalization has been fixed by $T_q(0) = 1_{id}$.

The momentum (P) is given by,

$$e^{iP} = \Lambda_Q(u = \frac{\pi}{2N}) = \left[\frac{g_q(0)\overline{g}_q(0)}{g_q(\frac{\pi}{2N})\overline{g}_q(\frac{\pi}{2N})}\right]^M \prod_{k=1}^{L} \frac{\sin(\frac{\pi}{2N} - v_k)}{\sin v_k} \tag{27}$$

Now we turn to the sectors $Q \neq 0$, and the symmetry under charge conjugation allows us to consider the sectors $Q = 1, 2, \ldots, (N-1)/2$ only. While we have not been able to obtain a proof like the one given above, one can show that, in the sector Q

$$\begin{aligned} a)\ \ A, B &\leq \frac{NM}{2} - Q, \qquad Q = 1, 2 \ldots \frac{N}{2} \\ b)\ \ A, B &\geq \frac{NM}{2} - \frac{N}{2} \end{aligned}$$

Following similar argument as before, we arrive at

$$A = B = \frac{NM}{2} - Q$$

The reader must be warned that this conclusion lacks rigor just like in the case of N odd. The factorization in terms of sines can be carried out without the appearance of a phase $(e^{2iu})^{\pm(B-A)}$. We assume this to be true also for the others Q sectors, and arrive at the general form

$$\Lambda_Q(u) = \left[\frac{g_q(0)\overline{g}_q(0)}{g_q(u)\overline{g}_q(u)}\right]^M \prod_{k=1}^{L} \frac{\sin(u - v_k)}{\sin v_k} \tag{28}$$

$$L = NM - 2Q, \qquad Q = 0, 1, \ldots \frac{N}{2}, \qquad \Lambda_{N-Q}(u) = \Lambda_Q(u)$$

From this, the eigenvalue of H is easily found to be

$$E = \sum_{k=1}^{L} \cot v_k \quad - 2M \sum_{j=1}^{N/2} \cot(\pi j/N) \tag{29}$$

The momentum (P) is given by,

$$e^{iP} = \Lambda_Q(u = \frac{\pi}{2N}) = \left[\frac{g_q(0)\overline{g}_q(0)}{g_q(\frac{\pi}{2N})\overline{g}_q(\frac{\pi}{2N})} \right]^M \prod_{k=1}^{L} \frac{\sin(\frac{\pi}{2N} - v_k)}{\sin v_k} \tag{30}$$

We shall use the set of functional equations for the eigenvalues of transfer matrices of Chiral Potts derived by Baxter, Bazhanov and Perk [2, 24]. This functional relation appears in reference [2] as Eq. (4.40) and has the following form,

$$\tilde{T}_{\bar{q}} = \sum_{m=0}^{N-1} c_{m,q} T^{-1}_{U^m q} T_q T^{-1}_{U^{m+1}q} X^{-m-1} \tag{31}$$

where $\tilde{T} = TS$, $\bar{q} = (a_{\bar{q}}, b_{\bar{q}}) = UR^{-1}(a_q, b_q)$, and

$$c_{m,q} = \left(\left(\prod_{j=0}^{m-1} \frac{b_p - \omega^{j+1} a_q}{a_p - \omega^j a_q} \right) \cdot \left(\prod_{j=m+1}^{N-1} \frac{\omega(a_p - \omega^j a_q)}{b_p - \omega^{j+1} a_q} \right) \cdot \left(\frac{N(b_q - b_p)(b_p - a_q)}{a_p b_p - \omega^m a_q b_q} \right) \right)^M$$

$$\begin{aligned} \tilde{T}_{\bar{q}} &= \sum_{s=0}^{\frac{N}{2}-1} \left(c_{2s,q} T^{-1}_{U^{2s}q} T_q T^{-1}_{U^{2s+1}q} X^{-2s-1} \right) + \sum_{s=0}^{\frac{N}{2}-1} \left(c_{2s+1,q} T^{-1}_{U^{2s+1}q} T_q T^{-1}_{U^{2s+2}q} X^{-2s-2} \right) \\ &= \sum_{s=0}^{\frac{N}{2}-1} \left(\frac{c_{2s,q}}{A_{s,q} A_{s,q'}} \cdot T^{-1}_{R^{2s}q} T_q T^{-1}_{R^{2s}(Uq)} X^{-1} + \frac{c_{2s+1,q}}{A_{s,q'} A_{s+1,q}} \cdot T^{-1}_{R^{2s}(Uq)} T_q T^{-1}_{R^{2s+2}q} X^{-1} \right) \end{aligned}$$

Define

$$p_{2s} = \frac{c_{2s,q}}{A_{s,q} A_{s,q'}}, \qquad d_{2s+1} = \frac{c_{2s+1,q}}{A_{s,q'} A_{s+1,q}}$$

The independent inverse factors of $T_{R^{2s}q}$ and $T_{R^{2s}(Uq)}$ are considered, and both sides of the above equation are multiplied by the appropriate common factor so as to get rid of inverses of transfer matrix. The appropriate factor is:

$$X \prod_{j=1}^{\frac{N}{2}} T_{R^{2j}q} \prod_{j=0}^{\frac{N}{2}-1} T_{R^{2j}(Uq)} \tag{32}$$

After multiplying we get

$$\begin{aligned} X \cdot \tilde{T}_{\bar{q}} \cdot \prod_{j=1}^{\frac{N}{2}} T_{R^{2j}q} T_{R^{2(j-1)}(Uq)} = \sum_{s=0}^{\frac{N}{2}-1} \Bigg(& p_{2s} \prod_{\substack{j=0 \\ j\neq s}}^{\frac{N}{2}} T_{R^{2j}q} \cdot \prod_{\substack{j=0 \\ j\neq s}}^{\frac{N}{2}-1} T_{R^{2j}(Uq)} + \\ & d_{2s+1} \prod_{\substack{j=0 \\ j\neq (s+1)}}^{\frac{N}{2}} T_{R^{2j}q} \cdot \prod_{\substack{j=0 \\ j\neq s}}^{\frac{N}{2}-1} T_{R^{2j}(Uq)} \Bigg) \end{aligned} \tag{33}$$

If one expresses $T_{p,q}$ and $T_{p,Uq}$ in terms of a complex parameter u, where $u = q - p$ as $k \to 0$,

$$\begin{aligned} T_q &\to T_q & T_{Uq} &\to T_{Uq}(u) \\ T_{R^{2k}q} &\to T_q(u + \tfrac{k\pi}{N}) & T_{R^{2k}(Uq)} &\to T_{(Uq)}(u + \frac{k\pi}{N}) \end{aligned} \tag{34}$$

with this parametrization we get

$$\begin{aligned} X \cdot \tilde{T}_{\bar{q}} \cdot \prod_{j=1}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \cdot \prod_{j=0}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) = \sum_{s=0}^{\frac{N}{2}-1} \Bigg(& p_{2s} \prod_{\substack{j=0 \\ j \neq s}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j \neq s}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) \\ & + d_{2s+1} \prod_{\substack{j=0 \\ j \neq s+1}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j \neq s}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) \Bigg) \end{aligned} \tag{35}$$

Let v be a zero of T_q, i.e., $T_q(v) = 0$, then whenever $u = v - \pi k/N$, $T_q(u + \pi k/N) = 0$. Thus for $u = v - \pi k/N \quad k \in \{1, 2, \ldots, \frac{N}{2} - 1\}$ all but two terms vanish.

$$p_{2k} \prod_{\substack{j=0 \\ j \neq k}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j \neq k}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) + d_{2k-1} \prod_{\substack{j=0 \\ j \neq k}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j \neq k-1}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) = 0 \tag{36}$$

Cancelling the common factors we get

$$p_{2k}(u) \cdot T_{Uq}(u + \frac{\pi k}{N} - \frac{\pi}{N}) + d_{2k-1}(u) \cdot T_{Uq}(u + \frac{\pi k}{N}) = 0 \tag{37}$$

whence

$$\frac{T_{Uq}(v)}{T_{Uq}(v - \frac{\pi}{N})} = -\frac{p_{2k}(v - \frac{\pi k}{N})}{d_{2k-1}(v - \frac{\pi k}{N})} \tag{38}$$

Recalling the expression for T_{Uq},

$$\prod_{j=1}^{L_{Uq}} \frac{\sin(v_i - \bar{v}_j)}{\sin(v_i - \bar{v}_j - \frac{\pi}{N})} \;=\; -\left(\frac{g_{_{Uq}}(v_i) \cdot \bar{g}_{_{Uq}}(v_i)}{g_{_{Uq}}(v_i - \frac{\pi}{N}) \cdot \bar{g}_{_{Uq}}(v_i - \frac{\pi}{N})} \right)^M \cdot \frac{p_{2k}(v_i - \frac{\pi k}{N})}{d_{2k-1}(v_i - \frac{\pi k}{N})} \tag{39}$$

The ratio of $g_{_{Uq}}$'s can be obtained as

$$\begin{aligned} g_{_{Uq}}(v_i) = \prod_{j=1}^{\frac{N}{2}-1} \sin(\frac{\pi j}{N} + \frac{\pi}{2N} + v_i), \qquad & g_{_{Uq}}(v_i - \frac{\pi}{N}) = \prod_{j=0}^{\frac{N}{2}-2} \sin(\frac{\pi j}{N} + \frac{\pi}{2N} + v_i) \\ \frac{g_{_{Uq}}(v_i)}{g_{_{Uq}}(v_i - \frac{\pi}{N})} = \frac{\sin(\frac{\pi}{2} - \frac{\pi}{N} + \frac{\pi}{2N} + v_i)}{\sin(0 + \frac{\pi}{2N} + v_i)} = \frac{\cos(v_i - \frac{\pi}{2N})}{\sin(v_i + \frac{\pi}{2N})} & \end{aligned} \tag{40}$$

Similarly the ratios of $\bar{g}_{_{Uq}}$ is found as,

$$\begin{aligned} \bar{g}_{_{Uq}}(v_i) = \prod_{j=1}^{\frac{N}{2}-1} \sin(\frac{\pi j}{N} - v_i), \qquad & \bar{g}_{_{Uq}}(v_i - \frac{\pi}{N}) = \prod_{j=2}^{\frac{N}{2}} \sin(\frac{\pi j}{N} - v_i) \\ \frac{\bar{g}_{_{Uq}}(v_i)}{\bar{g}_{_{Uq}}(v_i - \frac{\pi}{N})} = \frac{\sin(\frac{\pi}{N} - v_i)}{\sin(\frac{\pi}{2} - v_i)} = (-1) \cdot \frac{\sin(v_i + \frac{\pi}{N})}{\cos(v_i)} & \end{aligned} \tag{41}$$

Using these results for the ratios of g_{Uq} and those of (p_{2k}/d_{2k-1}) we finally get the Bethe equations [5]

$$\prod_{j=1}^{L_{Uq}} \frac{\sin(v_i - \bar{v}_j)}{\sin(v_i - \bar{v}_j - \frac{\pi}{N})} = (-1)^{M+1} \left[\frac{\sin 2(v_i - \frac{\pi}{2N})}{\sin(2v_i)}\right]^{2M} \tag{42}$$

Let $\bar{v}$ be a zero of $T_{_{Uq}}$, i.e., $T_{_{Uq}}(\bar{v}) = 0$, then whenever $u = \bar{v} - \pi k/N$, $T_q(u + \pi k/N) = 0$. For $u = \bar{v} - \pi k/N \quad k \in \{1, 2, \ldots, \frac{N}{2} - 1\}$ all but two terms vanish

$$p_{2k} \prod_{\substack{j=0 \\ j\neq k}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j\neq k}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) + d_{2k+1} \prod_{\substack{j=0 \\ j\neq k+1}}^{\frac{N}{2}} T_q(u + \frac{\pi j}{N}) \prod_{\substack{j=0 \\ j\neq k}}^{\frac{N}{2}-1} T_{Uq}(u + \frac{\pi j}{N}) = 0 \tag{43}$$

Cancelling the common factors we get

$$\begin{aligned} p_{2k}(u) \cdot T_q(u + \frac{\pi k}{N} + \frac{\pi}{N}) + d_{2k+1}(u) \cdot T_q(u + \frac{\pi k}{N}) = 0 \\ \frac{T_q(\bar{v})}{T_q(\bar{v} + \frac{\pi}{N})} = -\frac{p_{2k}(\bar{v} - \frac{\pi k}{N})}{d_{2k+1}(\bar{v} - \frac{\pi k}{N})} \end{aligned} \tag{44}$$

The ratios of $g_{_q}$'s can be obtained as

$$\begin{aligned} g_{_q}(\bar{v}_i) = \prod_{j=1}^{\frac{N}{2}} \sin(\frac{\pi j}{N} - \frac{\pi}{2N} + \bar{v}_i), \qquad g_{_q}(\bar{v}_i + \frac{\pi}{N}) = \prod_{j=2}^{\frac{N}{2}+1} \sin(\frac{\pi j}{N} - \frac{\pi}{2N} + \bar{v}_i) \\ \frac{g_{_q}(\bar{v}_i)}{g_{_q}(\bar{v}_i + \frac{\pi}{N})} = \frac{\sin(\frac{\pi}{N} - \frac{\pi}{2N} + \bar{v}_i)}{\sin(\frac{\pi}{N} + \frac{\pi}{N} + \bar{v}_i)} = \frac{\sin(\bar{v}_i + \frac{\pi}{2N})}{\cos(\bar{v}_i + \frac{\pi}{2N})} \end{aligned} \tag{45}$$

Similarly the ratios of $\overline{g}_{_q}$'s is found as,

$$\begin{aligned} \overline{g}_{_q}(\bar{v}_i) = \prod_{j=1}^{\frac{N}{2}} \sin(\frac{\pi j}{N} - \bar{v}_i), \qquad \overline{g}_{_q}(\bar{v}_i + \frac{\pi}{N}) = \prod_{j=0}^{\frac{N}{2}-1} \sin(\frac{\pi j}{N} - \bar{v}_i) \\ \frac{\overline{g}_{_q}(\bar{v}_i)}{\overline{g}_{_q}(\bar{v}_i + \frac{\pi}{N})} = \frac{\sin(\frac{\pi}{2} - \bar{v}_i)}{\sin(0 - v_i)} = (-1) \cdot \frac{\cos(\bar{v}_i)}{\sin(\bar{v}_i)} \end{aligned} \tag{46}$$

Using these results for the ratios of g_q and those of p_{2k}/d_{2k+1} we finally get

$$\prod_{j=1}^{L_q} \frac{\sin(\bar{v}_i - v_j)}{\sin(\bar{v}_i - v_j + \frac{\pi}{N})} = (-1)^{M+1}$$

In order to cast the BAE's for even case in a simpler (and standard) form, we make a change of variables.

$$v_j = i\lambda_j + \frac{\pi}{4N}, \qquad \bar{v}_j = i\bar{\lambda}_j - \frac{\pi}{4N} \tag{47}$$

The BAE's in terms of these new variables are

$$\prod_{k=1}^{L_{Uq}} \frac{\sinh(\lambda_j - \bar{\lambda}_k - i\gamma)}{\sinh(\lambda_j - \bar{\lambda}_k + i\gamma)} = (-1)^{M+1} \left[\frac{\sinh 2(\lambda_j + is\gamma)}{\sinh 2(\lambda_j - is\gamma)}\right]^{2M} \tag{48}$$

$$\prod_{k=1}^{L_q} \frac{\sinh(\bar{\lambda}_j - \lambda_k - i\gamma)}{\sinh(\bar{\lambda}_j - \lambda_k + i\gamma)} = (-1)^{M+1} \tag{49}$$

where $\gamma = \frac{\pi}{2N}$ and $s = \frac{1}{2}$. From the numerical study for even spin BAE-s, it was found that $\lambda_j s$ are related to one another. In fact

$$\forall \quad \lambda_j \quad \exists \quad \lambda_j + i\pi/2 \ \mathrm{mod}(\pi) \tag{50}$$

This allows us to group λ_j such that $\lambda_j \in [-\pi/4, \pi/4]$. Using transformation rules for the hyperbolic functions one can rewrite the expressions in terms of a new variable $\chi_j = 2\lambda_j$. The LHS of BAE(1) becomes

$$\begin{aligned}
&\prod_{k=1}^{\frac{L_{Uq}}{2}} \frac{\sinh(\lambda_j - \bar{\lambda}_k - i\gamma)}{\sinh(\lambda_j - \bar{\lambda}_k + i\gamma)} \cdot \frac{\sinh(\lambda_j - \bar{\lambda}_k - i\gamma - \frac{i\pi}{2})}{\sinh(\lambda_j - \bar{\lambda}_k + i\gamma - \frac{i\pi}{2})} \\
&= \prod_{k=1}^{\frac{L_{Uq}}{2}} \frac{\sinh 2(\lambda_j - \bar{\lambda}_k - i\gamma)}{\sinh 2(\lambda_j - \bar{\lambda}_k + i\gamma)} = \prod_{k=1}^{\frac{L_{Uq}}{2}} \frac{\sinh(\chi_j - \bar{\chi}_k - 2i\gamma)}{\sinh(\chi_j - \bar{\chi}_k + 2i\gamma)}
\end{aligned} \tag{51}$$

RHS of BAE(1) is rewritten in terms of variables χ_j

$$(-1)^{M+1} \left(\frac{\sinh(\chi_j + 2is\gamma)}{\sinh(\chi_j - 2is\gamma)} \right)^{2M} = (-1)^{M+1} \left(\frac{\sinh(\chi_j + i\gamma)}{\sinh(\chi_j - i\gamma)} \right)^{2M}, \quad \text{since } s = \frac{1}{2} \tag{52}$$

A similar transformation is done for BAE(2). Hence the BAE equations become

$$\prod_{k=1}^{\frac{L_{Uq}}{2}} \frac{\sinh(\chi_j - \bar{\chi}_k - 2i\gamma)}{\sinh(\chi_j - \bar{\chi}_k + 2i\gamma)} = (-1)^{M+1} \left(\frac{\sinh(\chi_j + i\gamma)}{\sinh(\chi_j - i\gamma)} \right)^{2M} \tag{53}$$

$$\prod_{k=1}^{\frac{L_q}{2}} \frac{\sinh(\bar{\chi}_j - \chi_k - 2i\gamma)}{\sinh(\bar{\chi}_j - \chi_k + 2i\gamma)} = (-1)^{M+1} \tag{54}$$

The eigenvalues $\Lambda_Q(u)$ of the transfer matrix are given in terms of these zeroes χ_js as [3],

$$\Lambda_Q(u) = \left[\frac{g_q(0)\overline{g}_q(0)}{g_q(u)\overline{g}_q(u)} \right]^M \prod_{k=1}^{\frac{L}{2}} \frac{\sin(2u - i\chi_k - \frac{\pi}{2N})}{\sin(i\chi_k + \frac{\pi}{2N})} \tag{55}$$

where,

$$g_q(u) = \prod_{j=1}^{N/2} \sin(\frac{\pi j}{N} - \frac{\pi}{2N} + u), \qquad \overline{g}_q(u) = \prod_{j=1}^{N/2} \sin(\frac{\pi j}{N} - u) \tag{56}$$

V. Study of finite size systems

The BAE in the present model differs from the standard form. The sign in front of γ on the right hand side (RHS) of BAE (1) is reversed. That is to say that the RHS is equal to the inverse of what usually is known to be the RHS. The second of the coupled pair, BAE (2) is even more striking. Though the LHS is still coupled, the RHS is independent of the spectral variable ! Understanding the real significance of these peculiarities can help enormously in solving the problem.

The transfer matrix for the FZM is constructed from the FZM Boltzmann weights (BW) as:

$$T_q^{\boldsymbol{n},\boldsymbol{n'}}(u) = T_{p,q}^{\boldsymbol{n},\boldsymbol{n'}}(u) = \prod_{k=1}^{M} \overline{W}_{pq}(n_k - n'_k|u) W_{pq}(n_k - n'_{k+1}|u). \tag{57}$$

where M is the number of sites in each row and periodic boundary condition is implied. These transfer matrices for different spectral variable u form a commuting family. Transfer matrix acts on vectors defined in terms of spin indices along a row (or diagonal) [3] or the spin configuration $\boldsymbol{n} = |n_1, n_2, \ldots n_M >$.

There is an associated transfer matrix $T_{p,Uq}$ which corresponds to a conjugate set of Boltzmann weights [4],

$$T_{Uq}^{\boldsymbol{n},\boldsymbol{n'}}(u) = T_{p,Uq}^{\boldsymbol{n},\boldsymbol{n'}}(u) = \prod_{k=1}^{M} \overline{W}_{pUq}(n_k - n'_k|u) W_{pUq}(n_k - n'_{k+1}|u). \tag{58}$$

The shift or translation operator $\widehat{S}$ is defined by its action on a state function or spin configuration $\boldsymbol{n} = |n_1, n_2, \ldots n_M >$.

$$\widehat{S}|n_1, n_2, n_3, \ldots n_M >= |n_M, n_1, n_2, \ldots n_{M-1} > \tag{59}$$

Momentum P is defined in terms of the shift operator as

$$e^{iP} = \widehat{S}^{-1}$$

The z-component of spin operator $\widehat{Z}_k$ and spin raising operator $\widehat{X}_k$ corresponding to a given lattice site (k) are defined by their action on a state function or spin configuration as $\boldsymbol{n} = |n_1, n_2, \ldots n_M >$, $n_k = 0, 1 \ldots N-1$.

$$\begin{aligned} \widehat{X}_k|n_1 \ldots n_k \ldots n_M > &= |n_1 \ldots n_k + 1 \ldots n_M > \quad \mathrm{mod}(N) \\ \widehat{Z}_k|n_1 \ldots n_k \ldots n_M > &= \omega^{n_k}|n_1 \ldots n_k \ldots n_M > \end{aligned}$$

The global spin raising operator is given by

$$\widehat{X} = \prod_{k=1}^{M} \widehat{X}_k$$

The spin Q is defined in terms of $\widehat{X}$ as

$$e^{iQ} = \widehat{X}^{-1}$$

$T_{p,q}(u)$ and $T_{p,Uq}(u)$ commute with $\widehat{S}$, $\widehat{Z}_k$ and $\widehat{X}$. Hence it is possible to make a spin and momentum sectorwise study of the problem.

The roots of the BAE are studied by computing the eigenvalues of the transfer matrices as meromorphic functions of $x = e^{\lambda}$. Since the transfer matrices with different spectral parameter λ commute, their eigenvectors are independent of the spectral parameter. Hence by taking a specific value of the spectral parameter one can determine the eigenvectors numerically by diagonalizing the finite size transfer matrix. From the definition of the eigenvalue equation one can express the eigenvalues as meromorphic functions of x, as the entries of transfer matrix are polynomials in x and the eigenvectors

are vectors with numerical (independent of x) entries.

However the problem being coupled, one needs to simultaneously diagonalize T_q and T_{Uq} or in other words find the eigenvalues corresponding to simultaneous eigenvectors of T_q and T_{Uq}. This fact by itself introduces a significantly higher level of difficulty over other numerical simulation of similar type (non coupled eqns.) e.g. Chiral Potts [25]. Coupled BAE and any simultaneous eigenvalue problem (eigenvalues corresponding to simultaneous eigenvectors) results in the same generic problem in computer algorithm. One has to develop efficient optimized codes for tackling this.

A detailed numerical study for chains of length $M \leq 8$ [3] was done. The main observations of this numerical study are as follows:

- 1-String with both parities: $(1, v)$, $v =$ (± 1)
- even length strings with positive parities: $(n, +)$, $n = 2, 4..., N$
- non-string solutions $Im(\lambda) \sim \pm\pi/3$

The Ferromagnetic ground state is a filled band of 2-strings, and the excitations consist of (1+), (1−) etc. The Antiferromagnetic ground state on the other hand is a filled band of (1, ±) and excitations are 2-strings with positive parity.

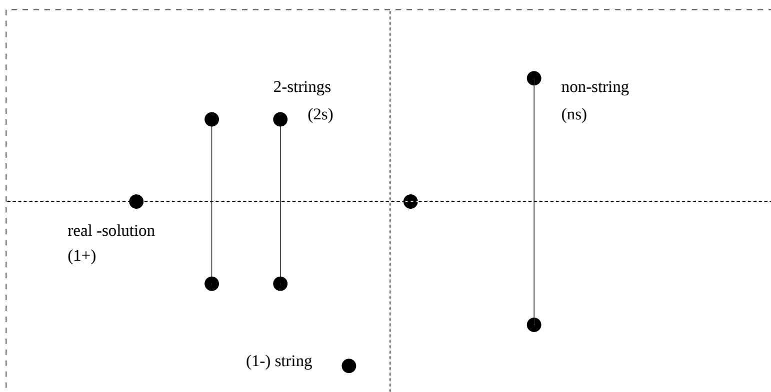

Figure 4: String and non-string solutions for BAE

It is remarkable that a good deal of insight into the nature of string solutions can be obtained from finite systems of rather small size. The 2-strings are easily identified as having imaginary parts close to $\pi/4$. One can also identify the real roots and roots with negative parity $Im(\lambda) = \pi/2$. There are roots whose imaginary parts are not well approximated by $\pi/2$ or $\pi/4$. These roots do not seem to systematically approach the 2-string value as $M \to \infty$. These roots are classified as non-strings. For space limitation $M = 2$ and $M = 3$ figures are presented in this paper. The imaginary parts of the non-string (ns) roots are

M=2	P=0	$-0.32198235\pi i$
		$0.32198235\pi i$
M=3	P=0	$-0.29730902\pi i$
		$0.29730902\pi i$
	P=1	$0.179912 - 0.330545\pi i$
		$0.179912 + 0.33054515\pi i$

	P=2	$-0.179912 - 0.330545\pi i$
		$-0.179912 + 0.33054515\pi i$
M=4	P=0	$-0.33780988\pi i$
		$0.33780988\pi i$
		$-0.348355 - 0.342016\pi i$
		$-0.348355 + 0.34201621\pi i$
		$0.348355 - 0.342016\pi i$
		$0.348355 + 0.34201621\pi i$
		$-0.28324874\pi i$
		$0.28324874\pi i$
	P=1	$-0.258619 - 0.38794057\pi i$
		$-0.258619 + 0.38794057\pi i$

The following tables show the classification of roots for spin-4, for lattice sizes 2, 3 and 4 in the $Q = 0$ sector. The first column shows the momentum P. The second column gives the roots of the BAE in the variable χ. The content or the type of root is identified in column three. Column four gives the corresponding integer that appears in the BAE. In column five the energy calculated for the corresponding eigenvalue for Transfer Matrix is given.

Table 1: Classification of roots and integers for $M = 2$

P	$\lambda_k = \ln x_k$	Content	I_k	Energy
	$-0.216337 - 0.227395\pi i$	(2s)	0.5	
	$-0.216337 + 0.22739469\pi i$			
	$0.216337 - 0.227395\pi i$	(2s)	-0.5	
0	$0.216337 + 0.22739469\pi i$			-6.24622
	-0.265319	(1+)	-0.5	
	$-0.32198235\pi i$	(ns)	0	
	$0.32198235\pi i$			
0	0.265319	(1+)	0.5	4.
	-0.440687	(1+)	$-1.$	
	0	(1+)	0	
	$-0.5\pi i$	(1−)	0	
1	0.440687	(1+)	1.	8.

Table 2: Classification of roots and integers for $M = 3$

P	$\lambda_k = \ln x_k$	Content	I_k	Energy
	$-0.325636 - 0.228897\pi i$	(2s)	1.	
	$-0.325636 + 0.22889664\pi i$			
	$-0.23986244\pi i$	(2s)	0	
	$0.23986244\pi i$			
	$0.325636 - 0.228897\pi i$	(2s)	$-1.$	
0	$0.325636 + 0.22889664\pi i$			-8.36945
	-0.388991	(1+)	$-1.$	
	$-0.267718\pi i$	(2s)	0	
	$0.267718\pi i$			
	$-0.29730902\pi i$	(ns)	0	
	$0.29730902\pi i$			
0	0.388991	(1+)	1.	-1.65686

Table 2 Contd.: Classification of roots and integers for $M = 3$

P	$\lambda_k = \ln x_k$	Content	I_k	Energy
	-0.570869	(1+)	-1.5	
	$-0.213538 - 0.5\pi i$	(1−)	0	
	-0.149008	(1+)	-0.5	
	0.0873277	(1+)	0.5	
	$0.423044 - 0.242196\pi i$	(2s)	-1.5	
0	$0.423044 + 0.24219619\pi i$			8.
	-0.768026	(1+)	$-2.$	
	-0.241758	(1+)	$-1.$	
	$-0.24949865\pi i$	(2s)	0	
	$0.24949865\pi i$			
	0.241758	(1+)	1.	
0	0.768026	(1+)	2.	4.48683
	$-0.353241 - 0.236642\pi i$	(2s)	1.	
	$-0.353241 + 0.23664231\pi i$			
	-0.0132479	(1+)	0	
	$0.179912 - 0.330545\pi i$	(ns)	0	
	$0.179912 + 0.33054515\pi i$			
1	0.359907	(1+)	1.	3.37155
	-0.540092	(1+)	-1.5	
	-0.129032	(1+)	-0.5	
	$-0.0725776 - 0.5\pi i$	(1−)	0	
	$0.117714 - 0.250431\pi i$	(2s)	-0.5	
	$0.117714 + 0.25043057\pi i$			
1	0.506275	(1+)	1.5	3.10102
	-0.692847	(1+)	$-2.$	
	$-0.209487 - 0.250419\pi i$	(2s)	1.	
	$-0.209487 + 0.25041875\pi i$			
	0.0217873	(1+)	0	
	0.2683	(1+)	1.	
1	0.821734	(1+)	2.	6.82843
	-0.856212	(1+)	$-2.$	
	-0.287304	(1+)	$-1.$	
	-0.0386368	(1+)	0	
	0.185678	(1+)	1.	
	$0.375937 - 0.5\pi i$	(1−)	-0.5	
1	0.620538	(1+)	2.	12.899
	-0.885356	(1+)	$-2.$	
	-0.304098	(1+)	$-1.$	
	-0.0535889	(1+)	0	
	0.16349	(1+)	1.	
	$0.539776 - 0.247284\pi i$	(2s)	$-1.$	
1	$0.539776 + 0.24728422\pi i$			11.4569
	-0.359907	(1+)	$-1.$	
	$-0.179912 - 0.330545\pi i$	(ns)	0	
	$-0.179912 + 0.33054515\pi i$			
	0.0132479	(1+)	0	
	$0.353241 - 0.236642\pi i$	(2s)	$-1.$	
2	$0.353241 + 0.23664231\pi i$			3.37155

Table 2 Contd.: Classification of roots and integers for $M = 3$

P	$\lambda_k = \ln x_k$	Content	I_k	Energy
	-0.506275	(1+)	-1.5	
	$-0.117714 - 0.250431\pi i$	(2s)	0.5	
	$-0.117714 + 0.25043057\pi i$			
	$0.0725776 - 0.5\pi i$	(1−)	0	
	0.129032	(1+)	0.5	
2	0.540092	(1+)	1.5	3.10102
	$-0.539776 - 0.247284\pi i$	(2s)	1.	
	$-0.539776 + 0.24728422\pi i$			
	-0.16349	(1+)	$-1.$	
	0.0535889	(1+)	0	
	0.304098	(1+)	1.	
2	0.885356	(1+)	2.	11.4569
	-0.620538	(1+)	$-2.$	
	$-0.375937 - 0.5\pi i$	(1−)	0.5	
	-0.185678	(1+)	$-1.$	
	0.0386368	(1+)	0	
	0.287304	(1+)	1.	
2	0.856212	(1+)	2.	12.899
	-0.821734	(1+)	$-2.$	
	-0.2683	(1+)	$-1.$	
	-0.0217873	(1+)	0	
	$0.209487 - 0.250419\pi i$	(2s)	$-1.$	
	$0.209487 + 0.25041875\pi i$			
2	0.692847	(1+)	2.	6.82843

VI. Free Energy in the Ferromagnetic case for N even

From the numerical study one can identify that the Ferromagnetic (FM) ground state corresponds to a filled band of $N/2$ strings of positive parity for T_q and a filled band of 1-string of negative parity for T_{Uq}. This vector always falls in the $P = 0$ sector as is expected. A further study upto 6 sites reveals that this remains true.

The LHS of the first of Bethe Ansatz Equations, BAE(1), is given by:

$$\prod_{k=1}^{\frac{L_{Uq}}{2}} \frac{\sinh(\chi_j - \bar{\chi}_k - 2i\gamma)}{\sinh(\chi_j - \bar{\chi}_k + 2i\gamma)} = (-1)^{M+1} \left[\frac{\sinh(\chi_j + \gamma i)}{\sinh(\chi_j - \gamma i)}\right]^{2M} \tag{60}$$

For the Ferromagnetic case we made the assumption that the ground state corresponds to $N/2$ strings with positive parity for T_q and 1-strings of negative parity for T_{Uq}.

$$\chi_{\alpha,v}^{n,l} = \chi_\alpha^n + 2\gamma(n+1-2l)i \tag{61}$$

$$\bar{\chi}_{\alpha,v}^{n,l} = \bar{\chi}_\alpha^n - \frac{i\pi}{2} \tag{62}$$

Define x and $\bar{x}$ by $\chi = 2\gamma x$ and $\bar{\chi} = 2\gamma\bar{x}$. If $M^{(n)}$ denotes the number of n-strings, we get for the left hand side (LHS) of the BAE(1)

$$\prod_{l=1}^{n} \frac{\sinh 2\gamma(x_\alpha^n - \bar{x}_\beta + (n+1-2l)i - i - p_0 i)}{\sinh 2\gamma(x_\alpha^n - \bar{x}_\beta + (n+1-2l)i + i - p_0 i)} = \frac{\sinh 2\gamma(x_\alpha^n - \bar{x}_\beta - ni - p_0 i)}{\sinh 2\gamma(x_\alpha^n - \bar{x}_\beta + ni - p_0 i)}$$

Taking product over the string elements of χ_α the first BAE becomes decoupled and is given in terms of the variables for T_q alone.

$$(-1)^{(M+1)n}\left[\prod_{l=1}^{n}\frac{\sinh 2\gamma(x_\alpha^n+(n+1-2l+\frac{1}{2})i)}{\sinh 2\gamma(x_\alpha^n-(n+1-2l+\frac{1}{2})i)}\right]^{2M}=(-1)^{M+1} \tag{63}$$

After multiplying for the elements of a string, the second BAE becomes

$$\prod_{k=1}^{\frac{L_q}{2}}\prod_{l=1}^{n}\frac{\sinh 2\gamma(\bar{x}_\alpha-x_\beta^n-(n+1-2l)i-i-p_0 i)}{\sinh 2\gamma(\bar{x}_\alpha-x_\beta^n-(n+1-2l)i+i-p_0 i)}=(-1)^{(M+1)+\frac{nL_q}{2}} \tag{64}$$

Thus we have only one set of BAE which involves the zeros of the transfer matrix T_q. From the above equation taking natural logarithm of both sides we get

$$2M\sum_{l=1}^{n} i\cdot\ln\left(\frac{\sinh(\chi_\alpha+2\gamma(n+1-2l+\frac{1}{2}))}{\sinh(\chi_\alpha-a\gamma(n+1-2l+\frac{1}{2}))}\right)=\pi I_\alpha$$

Defining the density of string centers for the zeros of T_q by

$$\rho(\chi)=\lim_{M\to\infty}\frac{1}{M(\chi_{k+1}-\chi_k)} \tag{65}$$

we get,

$$\rho(\chi)=\frac{1}{\pi}\Theta^{(1)'}_{(\frac{N}{2},+)}(\chi) \tag{66}$$

where

$$\Theta^{(1)}_{(\frac{N}{2},+)}(\chi)\doteq\sum_{l=1}^{n}2\phi(\chi,n+\frac{1}{2}-2l,+)$$

and prime on $\Theta^{(1)}_{(\frac{N}{2},+)}(\chi)$ denotes differentiation with respect to the variable χ. Here the function ϕ, as defined by Takahashi and Suzuki [26] is,

$$\phi(x,n,v)\doteq i\cdot\ln(g(x,n,v)) \quad ; \qquad g(x,n,v)\doteq\frac{\sinh 2\gamma(x+ni+p_0 i)}{\sinh 2\gamma(x-ni+p_0 i)}$$

Evaluating the sum over l in the Fourier space,

$$\tilde{\Theta}^{(1)'}_{(\frac{N}{2},+)}(k) = 2\pi\frac{\sinh(\frac{\pi k}{2}-\frac{\pi k}{2N})}{\sinh(\frac{\pi k}{N})} \tag{67}$$

$$\tilde{\rho}(k) = 4\frac{\sinh(\frac{\pi k}{2}-\frac{\pi k}{2N})}{\sinh(\frac{\pi k}{N})} \tag{68}$$

By inverse Fourier transform we get

$$\rho(\chi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dk e^{-ik\chi}\tilde{\rho}(k)$$

The free energy for the Ferromagnetic ground state is defined as

$$f_0(u)\doteq\lim_{M\to\infty}\left(-\frac{1}{M}\ln\Lambda_0(u)\right)=\prod_{\alpha=1}^{M\frac{N}{2}}\prod_{l=1}^{\frac{N}{2}}\frac{\sin(2u-i\chi_\alpha+x(l))}{\sin(i\chi_\alpha-x(l))}$$

Replacing the summation by an integral over the symmetrically placed string centers,

$$f_0(u) = -\frac{1}{2}\int_{-\infty}^{\infty} d\chi\rho(\chi)\sum_{l=1}^{\frac{N}{2}} \ln\left(\frac{\cosh(2\chi)-\cos(4u+2x(l))}{\cosh(2\chi)-\cos(2x(l))}\right) \tag{69}$$

Transforming to the Fourier space and utilizing the expression and properties of $\tilde{\rho}(k)$ we get

$$f_0(u) = \int_{-\infty}^{\infty} \frac{4\ \ dk}{k}\frac{\sinh(k\pi-\frac{k\pi}{N})\sinh(2ku)\sinh(2ku-k\pi-\frac{k\pi}{N})}{\sinh^2(\frac{2k\pi}{N})} \tag{70}$$

VII. Excitation on FM ground state

We have seen in the previous section that the FM ground state is given by a filled band of $(2s)$ strings. Consider the $Z_{(2s)}(\chi)$ function. In the general case it should look like

$$Z_{(2s)}(\chi) = \frac{1}{2\pi}\Theta^{(1)}_{(2s)}(\chi) - \frac{1}{2\pi M}\sum_k\sum_{\beta=1}^{M^{(k)}}\Theta^{(2)}_{(2s,k)}(\chi-\chi^k_\beta) \tag{71}$$

The density of $(2s)$ vacancies is given by

$$\sigma_{(2s)}(\chi) \doteq -Z'_{(2s)}(\chi) \tag{72}$$

The vacancy density and the density of (2s) particles is related by

$$\sigma_{(2s)}(\chi) = \rho_{(2s)}(\chi) + \frac{1}{M}\sum_{\beta=1}^{M_h^{(2s)}}\delta\left(\chi-\chi^{(2s)h}_\beta\right) \tag{73}$$

where $\chi^{(2s)h}_\beta$ are the position of the holes.
Thus

$$-\sigma_{(2s)}(\chi) = \frac{1}{2\pi}\Theta^{(1)'}_{(2s)}(\chi) - \frac{1}{2\pi M}\sum_{k\neq(2s)}\sum_{\beta=1}^{M^{(k)}}\Theta^{(2)'}_{(2s,k)}(\chi-\chi^k_\beta) -$$

$$\frac{1}{2\pi}\int\Theta^{(2)'}_{(2s,2s)}(\chi-\mu)d\mu + \frac{1}{2\pi M}\sum_{h=1}^{M_h^{(2s)}}\Theta^{(2)'}_{(2s,2s)}(\chi-\chi^{(2s)h}_\beta) \tag{74}$$

The above equation can be interpreted as a collection of terms contributing to $(2s)$-ground state, $(2s)$- holes and excited particles $\sigma_{(2s)} = \sigma^{(0)}_{(2s)} + \sigma^{(h)}_{(2s)} + \sum_j\sigma^{(j)}_{(2s)}$ where $\sigma^{(0)}_{(2s)}$ is the same as $\rho_{(2s)}$ of the last section.

The expressions for energy (E) and momentum (P) are

$$E = \sum_{k=1}^{\frac{L}{2}}\cot\left(i\chi_k+\frac{\pi}{2N}\right) - 2M\sum_{k=1}^{\frac{N}{2}}\cot\left(\frac{\pi k}{N}\right) \tag{75}$$

$$e^{iP} = \prod_{k=1}^{\frac{L}{2}}\frac{\sinh\left(\chi_k+\frac{i\pi}{2N}\right)}{\sinh\left(\chi_k-\frac{i\pi}{2N}\right)} \tag{76}$$

We obtain the energy of a state designated by a given set of strings

$$\begin{aligned} E &= \sum_{\substack{k \\ strings}} \sum_{\beta=1}^{M^{(k)}} \epsilon_k(\chi_\beta^{(k)}) \\ &= \int d\chi \sigma_{(2s)}(\chi)\epsilon_{(2s)}(\chi) - \sum_{\beta=1}^{M_h^{(2s)}} \epsilon_{(2s)}\left(\chi_\beta^{(2s)h}\right) + \sum_{\beta \neq k} \sum_{\beta=1}^{M^{(k)}} \epsilon\left(\chi_\beta^{(k)}\right) \end{aligned} \quad (77)$$

The bare energies for n-string with parity v is easily obtained

$$\begin{aligned} \epsilon_{(n,v)}(\chi_\alpha) &= \sum_{k=1}^{n} \cot\left(i\chi_{k,\alpha}^{(n,v)} + \frac{\pi}{2N}\right) \\ &= \sum_{k=1}^{n} \cot\left(i\chi_\alpha^{(n,v)} - 2\gamma(n+1-2k) - \frac{\pi}{4}(1-v) + \frac{\pi}{2N}\right) \\ &= \sum_{k=1}^{n} \tan\left(\frac{\pi}{2} - i\chi_\alpha^{(n,v)} + 2\gamma(n+1-2k) + \frac{\pi}{4}(1-v) - \frac{\pi}{2N}\right) \end{aligned}$$

One can separate the real and imaginary parts of this expression. This helps in determining whether we require additional constraints on rapidities χ_j to ensure reality of the total energy.

Numerical study showed that there exist several spurious solutions and only a subset of them, corresponding to a specific choice of counting numbers $I_\alpha^{(j)}$, is admissible. Non-string solutions exist, however they are not as numerous. From the numerical study we make the assumption that the elementary excitations over the FM (a sea of $(2s)$-strings) are (a) a pair of (1+) strings , and (b) (1+) and (1−) strings.

The FM ground state is a filled band of $(2s)$ strings. The density of ground state energy is

$$e_0 = \lim_{M\to\infty} \frac{E_0}{M} = \int d\chi \rho_{(2s)}(\chi)\epsilon_{(2s)}(\lambda) - 2\sum_{k=1}^{\frac{N-1}{2}} \cot\left(\frac{\pi k}{N}\right) \quad (78)$$

The observed correlation between the integers suggest that the rapidities corresponding to $(2s)$ and (a) ought to be connected allowing cancellation of the imaginary part of the total energy.

$$Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(a)}\right] \quad (79)$$

It can be shown that, $Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(a)}\right]$ and $Re\left[\epsilon_{(2s)}\right] = -Re\left[\epsilon_{(a)}\right]$.

$$Re\left[\epsilon_{(a)}(\chi)\right] = \frac{4}{\cosh(4\chi)} \quad (80)$$

Similar argument holds for (b)-type excitations, where $Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(b)}\right]$ and $Re\left[\epsilon_{(2s)}\right] = -Re\left[\epsilon_{(b)}\right]$.

$$Re\left[\epsilon_{(b)}(\chi)\right] = \frac{4}{\cosh(4\chi)} \quad (81)$$

One should note that the dressed energy and bare energy are equal since the function coupling the ground state density to excited states $\Theta_{(j,k)}^{(2)}$ is zero for $j = (2s)$. Thus we

arrive at

$$E = E_0 + \sum_{\beta=1}^{M^{(a)}} 2\epsilon_{(a)}\left(\chi_\beta^{(a)}\right) + \sum_{\beta=1}^{M^{(b)}} 2\epsilon_{(b)}\left(\chi_\beta^{(b)}\right) \tag{82}$$

where E_0 is the ground state energy.

We now turn to the calculation of momentum. The momentum associated with a string of length j and parity v is found to be

$$\begin{aligned} p_{(1+)}(\chi) &= -\frac{1}{2}\Theta^{(1)}_{(1+)}(\chi) \\ p_{(j)}(\chi) &= -\frac{1}{2}\Theta^{(1)}_{(j)}(\chi) \quad for \;\; j \neq (1+) \end{aligned} \tag{83}$$

whence

$$p_{(a)}(\chi) = p_{(b)}(\chi) = 2\arctan\left(\tanh\left(\frac{\chi}{2}\right)\right) + \pi \tag{84}$$

These expressions are similar to the ones for non-string excitations in the Anti-Ferromagnetic case for odd spin FZM. We get the dispersion relations for elementary excitations over the Ferromagnetic ground state as,

$$\begin{aligned} \epsilon_{(a)}(p) &= 4\sin\left(\frac{p}{2}\right) \\ \epsilon_{(b)}(p) &= 4\sin\left(\frac{p}{2}\right) \end{aligned} \tag{85}$$

VIII. Free Energy in the Anti-Ferromagnetic case for N even

From the numerical study for finite lattices it was apparent that the Anti-Ferromagnetic (AFM) ground-state corresponds to a filled band of real roots for both T_q and T_{Uq}. In other words the AFM ground state is a filled sea of $(1,+)$ strings for both families of transfer matrices.

The AFM ground state corresponds to real roots for both families of transfer matrices. Hence we consider the natural logarithm of both sides of BAE-s. From BAE(1) we get,

$$\sum_{k=1}^{L_{Uq}/2} i\ln\left(\frac{\sinh(\chi_j - \bar{\chi}_k - 2i\gamma)}{\sinh(\chi_j - \bar{\chi}_k + 2i\gamma)}\right) = 2M \cdot i\ln\left(\frac{\sinh(\chi_j + i\gamma)}{\sinh(\chi_j - i\gamma)}\right) + 2\pi I_j \tag{86}$$

with standard definition of $\rho_1(\chi)$ and $\rho_2(\chi)$,

$$\begin{aligned} \rho_1(\chi_\alpha) &= \frac{1}{\pi}\Theta_1^{(1)'}(\chi_\alpha) - \frac{1}{2M\pi}\sum_{\beta=1}^{\bar{M}} \Theta_1^{(2)'}(\chi_\alpha - \bar{\chi}_\beta) \\ &= \frac{1}{\pi}\Theta_1^{(1)'}(\chi_\alpha) - \frac{1}{2M\pi}\int_{-\infty}^{+\infty} d\bar{\mu}\,\Theta_1^{(2)'}(\chi - \bar{\mu})\rho_1(\bar{\mu}) \end{aligned} \tag{87}$$

From BAE(2) we get,

$$\sum_{k=1}^{L_q/2} i\ln\left(\frac{\sinh(\bar{\chi}_j - \chi_k - 2i\gamma)}{\sinh(\bar{\chi}_j - \chi_k + 2i\gamma)}\right) = 2\pi\bar{I}_j \tag{88}$$

In the continuum limit $M \to \infty$ we get,

$$\rho_2(\bar{\chi}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\mu \Theta_2^{(2)'}(\bar{\chi}_\alpha - \mu)\rho_1(\mu) \tag{89}$$

The above pair of BAE is solved as before by the Fourier transform method.

$$\tilde{\rho}_1(k) = \frac{1}{\pi}\tilde{\Theta}_1^{(1)'}(k) - \frac{1}{2\pi}\tilde{\Theta}_1^{(2)'}(k)\tilde{\rho}_2(k) \tag{90}$$

$$\tilde{\rho}_2(k) = \frac{1}{2\pi}\tilde{\Theta}_2^{(2)'}(k)\tilde{\rho}_1(k) \tag{91}$$

From the above two equations,

$$\tilde{\rho}_1(k) = \frac{\sinh(\frac{\pi k}{2}) \cdot \sinh(\frac{\pi k}{2} - \frac{\pi k}{2N})}{\sinh(\frac{\pi k}{N}) \cdot \sinh(\pi k + \frac{\pi k}{N})} \tag{92}$$

Following a similar procedure as shown in detail in the Ferromagnetic case, the free energy for the Antiferromagnetic case is obtained as,

$$f = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{k}\frac{\tilde{\rho}_1(2k)}{\sinh(\pi k)}\left[\cosh k(\pi - 4u - \frac{\pi}{N}) - \cosh k(\pi - \frac{\pi}{N})\right] \tag{93}$$

Substituting for $\tilde{\rho}_1(2k)$

$$f = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{k}\frac{\sinh(\pi k - \frac{\pi k}{N})}{\sinh(\frac{2\pi k}{N}) \cdot \sinh(2\pi k + \frac{2\pi k}{N})} \cdot \cosh k(\pi - 4u - \frac{\pi}{N}) - \int_{-\infty}^{+\infty} \frac{dk}{k}\frac{\sinh(\pi k - \frac{\pi k}{N})}{\sinh(\frac{2\pi k}{N}) \cdot \sinh(2\pi k + \frac{2\pi k}{N})} \cdot \cosh k(\pi - \frac{\pi}{N}) \tag{94}$$

IX. Excitation on AFM ground state

In the case of AFM, the ground state is a filled band of (1+) . The results of previous section suggest that excitations appear as $(2s)$ strings when a hole is created in the (1+) sea. Consider the $Z_{(1+)}(\chi)$ function. In the general case it should look like

$$Z_{(1+)}(\chi) = \frac{1}{2\pi}\Theta_{(1+)}^{(1)}(\chi) - \frac{1}{2\pi M}\sum_k \sum_{\beta=1}^{M^{(k)}} \Theta_{(1+,k)}^{(2)}(\chi - \chi_\beta^k) \tag{95}$$

The density of (1+) vacancies is given by

$$\sigma_{(1+)}(\chi) \doteq -Z'_{(1+)}(\chi) \tag{96}$$

The vacancy density $\sigma_{(1+)}(\chi)$ and the density of (1+) particles $\rho_{(1+)}(\chi)$ is related by

$$\sigma_{(1+)}(\chi) = \rho_{(1+)}(\chi) + \frac{1}{M}\sum_{\beta=1}^{M_h^{(1+)}} \delta\left(\chi - \chi_\beta^{(1+)h}\right) \tag{97}$$

where $\chi_\beta^{(1+)h}$ are the position of the holes.
Thus

$$-\sigma_{(1+)}(\chi) = \frac{1}{2\pi}\Theta_{(1+)}^{(1)'}(\chi) - \frac{1}{2\pi M}\sum_{k\neq(1+)} \sum_{\beta=1}^{M^{(k)}} \Theta_{(1+,k)}^{(2)'}(\chi - \chi_\beta^k) - \frac{1}{2\pi}\int \Theta_{(1+,1+)}^{(2)'}(\chi - \mu)d\mu + \frac{1}{2\pi M}\sum_{\beta=1}^{M_h^{(1+)}} \Theta_{(1+,1+)}^{(2)'}(\chi - \chi_\beta^{(1+)h}) \tag{98}$$

The above equation can be interpreted as a collection of terms contributing to (1+) ground state, (1+) holes and excited particles over the ground state sea of (1+) string. $\sigma_{(1+)} = \sigma^{(0)}_{(1+)} + \sigma^{(h)}_{(1+)} + \sum_j \sigma^{(j)}_{(1+)}$, where $\sigma^{(0)}_{(1+)}$ is the same as $\rho_{(1+)}$ of the last section.

We obtain the energy of a state designated by a given set of strings $\{k\}$, having $M^{(k)}$ strings of type (k) with string centers at $\chi^{(k)}_\beta$,

$$\begin{aligned} E &= \sum_{\substack{k \\ strings}} \sum_{\beta=1}^{M^{(k)}} \epsilon_k(\chi^{(k)}_\beta) \\ &= \int d\chi \sigma_{(1+)}(\chi)\epsilon_{(1+)}(\chi) - \sum_{\beta=1}^{M_h^{(1+)}} \epsilon_{(1+)}\left(\chi_\beta^{(1+)h}\right) + \sum_{k\neq(1+)} \sum_{\beta=1}^{M^{(k)}} \epsilon\left(\chi^{(k)}_\beta\right) \end{aligned}$$

From the numerical study we can make the assumption that the elementary excitations over the AFM ground state, which is a sea of (1+)-strings, is given by a set of $(2s)$ strings.

The density of ground state energy for the AFM case is given by,

$$e_0 = \lim_{M\to\infty} \frac{E_0}{M} = \int d\chi \rho_{(1+)}(\chi)\epsilon_{(1+)}(\lambda) - 2\sum_{k=1}^{\frac{N-1}{2}} \cot\left(\frac{\pi k}{N}\right) \tag{99}$$

Total energy is real since the imaginary part of the energy contribution from holes cancel the imaginary part of the energy of excitation, i.e., $Im\left[\epsilon_{(2s)}\right] = Im\left[\epsilon_{(1+)}\right]$. The real parts are given by,

$$Re\left[\epsilon_{(2s)}(\chi)\right] = Re\left[\epsilon_{(1+)h}(\chi)\right] \tag{100}$$

In this case also, the dressed energy equals the bare energy. As before, we will denote by $\epsilon_{(2s)}\left(\chi^{(2s)}_\beta\right)$ its real part. The total energy is given by,

$$E = E_0 + \sum_{\beta=1}^{M^{(2s)}} 2\epsilon_{(2s)}\left(\chi^{(2s)}_\beta\right) \tag{101}$$

where E_0 is the ground state energy.

We now turn to the calculation of momentum. The momentum associated with a string of length j and parity v is found to be,

$$p_{(1+)}(\chi) = -\frac{1}{2}\Theta^{(1)}_{(1+)}(\chi) \tag{102}$$

$$p_{(j)}(\chi) = -\frac{1}{2}\Theta^{(1)}_{(j)}(\chi), \qquad \text{for } j \neq (1+) \tag{103}$$

whence

$$p_{(2s)}(\chi) = 2\arctan\left(\tanh\left(\frac{\chi}{2}\right)\right) \tag{104}$$

If one keeps in mind the fact that the correlation of χs demand an additional π for the creation of (1+) hole, we can derive the dispersion relation

$$\epsilon_{(2s)}(p) = 4\sin\left(\frac{p}{2}\right) \tag{105}$$

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